

Lie symmetry properties of nonlinear reaction-diffusion equations with gradient-dependent diffusivity

Roman Cherniha^{a,b1} John R. King^{a2} and Sergii Kovalenko^{c3}

^a *School of Mathematical Sciences, University of Nottingham,
University Park, Nottingham, NG7 2RD, UK*

^b *Institute of Mathematics, Ukrainian National Academy of Sciences,
3, Tereshchenkivs'ka Street, Kyiv 01601, Ukraine*

^c *Department of Physics, Poltava National Technical University,
24, Pershotravnevyi Prospekt, Poltava 36011, Ukraine*

Abstract

Complete descriptions of the Lie symmetries of a class of nonlinear reaction-diffusion equations with gradient-dependent diffusivity in one and two space dimensions are obtained. A surprisingly rich set of Lie symmetry algebras depending on the form of diffusivity and source (sink) in the equations is derived. It is established that there exists a subclass in 1-D space admitting an infinite-dimensional Lie algebra of invariance so that it is linearisable. A special power-law diffusivity with a fixed exponent, which leads to wider Lie invariance of the equations in question in 2-D space, is also derived. However, it is shown that the diffusion equation without a source term (which often arises in applications and is sometimes called the Perona-Malik equation) possesses no rich variety of Lie symmetries depending on the form of gradient-dependent diffusivity. The results of the Lie symmetry classification for the reduction to lower dimensionality, and a search for exact solutions of the nonlinear 2-D equation with power-law diffusivity, also are included.

¹E-mail: r.m.cherniha@gmail.com

²E-mail: John.King@nottingham.ac.uk

³E-mail: kovss82@gmail.com

1 Introduction

Nonlinear diffusion equations of the form

$$u_t = \operatorname{div} (D(u) \nabla u) \quad (1)$$

(hereafter $u = u(t, \bar{x})$ is an unknown function, $\bar{x} = (x_1, \dots, x_n)$, D is an arbitrary nonnegative function of its argument, $\nabla u \equiv (u_{x_1}, \dots, u_{x_n})$ and the subscripts $t, x_1 \dots x_n$, denote differentiation with respect to these variables) have an enormous number of diverse applications and have accordingly been widely studied by Lie symmetry method for a long time (see e.g. [19] for $n = 1$, [10, 12] for $n = 2, 3$).

An alternative class with similar status can be written in the form

$$u_t = \operatorname{div} (D(\Omega) \nabla u), \quad \Omega = |\nabla u|^2, \quad (2)$$

and also has a significant number of applications (some of which we note below) but has been much less studied from symmetry point of view (excepting the case $n = 1$). An observation relating to the similar standing of (1) and (2) is that the later corresponds to a gradient flow

$$\frac{\partial u}{\partial t} = - \frac{\delta L}{\delta u}$$

involving the variational derivative of

$$L = \frac{1}{2} \int \Psi(|\nabla u|^2) d\bar{x}, \quad D(\Omega) = \Psi'(\Omega), \quad d\bar{x} \equiv dx_1 \dots dx_n,$$

while the former corresponds to

$$\frac{\partial u}{\partial t} = - \frac{1}{\Psi'(u)} \frac{\delta L}{\delta u}$$

with

$$L = \frac{1}{2} \int |\nabla \Psi(u)|^2 d\bar{x}, \quad D(u) = \Psi'(u).$$

A third equation of similar status is

$$w_t = \Psi'(\Delta w), \quad \Delta \equiv \nabla^2 \quad (3)$$

which is reduced to (1), now with $D(u) = \Psi''$ by the non-local substitution $u = \Delta w$: this relationship is a non-local one, so the two may have different Lie symmetry properties. Taking into account this substitution and making the calculations, (1) and (3) can be written as the system

$$\frac{\partial u}{\partial t} = \frac{\delta L}{\delta w}, \quad \frac{\partial w}{\partial t} = \frac{\delta L}{\delta u},$$

where

$$L = \int \Psi(u) d\bar{x} = \int \Psi(\Delta w) d\bar{x}.$$

Finally, we in this case arrive formally at the equation

$$\frac{dL}{dt} = - \int \left| \nabla \frac{\delta L}{\delta u} \right|^2 d\bar{x}.$$

In one dimension equations (1), (2) and (3) are non-locally equivalent and we have the hierarchy (replacing u by v in the second equation)

$$u_t = (\Psi'(u))_{xx}, \quad v_t = (\Psi'(v_x))_x, \quad w_t = \Psi'(w_{xx})$$

with $u = v_x$ and $v = w_x$. Lie symmetry properties of this hierarchy have been studied by [1, 2, 5]. However, when source (sink) terms are included, as we do below, the corresponding reaction-diffusion equations are inequivalent even in one dimension.

It should be noted that equation (2) for $n = 2$ was proposed by Perona and Malik in [21] as an edge-enhancement model in image processing and this represent one of the most prominent applications of (2), stimulating a great deal of interest in the image processing community during the last two decades. It is commonly proposed that the Perona-Malik equation provides a valuable algorithm for image segmentation, noise removal, edge detection, and image enhancement (see, e.g., [13]). In (2), the unknown function $u = u(t, x_1, x_2)$ describes the time evolution from an original image, while the diffusion coefficient D (hereafter $D_\Omega \neq 0$) is a nonnegative function of the magnitude of local image gradient $\nabla u \equiv (u_{x_1}, u_{x_2})$. In [21], the authors proposed two forms for the diffusion coefficient D , namely

$$D(\Omega) = \exp\left(-\frac{\Omega}{D_0}\right), \quad D(\Omega) = \left(1 + \frac{\Omega}{D_0}\right)^{-1} \quad (4)$$

(here D_0 is a positive constant), which are the most common in image modelling. Equation (2) with the power-law diffusion coefficient

$$D(\Omega) = \Omega^m, \quad m > 0 \quad (5)$$

has also been used in modelling turbulent gas flows: see [4], for example.

Rigorous mathematical results for the Perona-Malik equation have been obtained mostly under the one-dimensional restriction, i.e.

$$u_t = (D(u_x)u_x)_x \quad (6)$$

and relevant references are given in [22] (see also the earlier papers [11, 14]). Equation (6) is sometimes called the nonlinear filtration equation and is applied in modelling non-Newtonian flows of fluids.

Here we concentrate ourselves on the investigation of (2) and its generalisations by means of the Lie symmetry method. Because their Lie symmetry properties essentially depend on the space dimensionality, we examine separately the cases $n = 1$ and $n = 2$. The Lie symmetries of (6) were completely described many years ago [2]. However, to the best of our knowledge, there are no rigorous mathematical results concerning the Lie symmetry of (2) in higher dimensions. Moreover, its extensions involving a source (sink) naturally arise in applications (see, for example, [20], [13]). Thus, one may consider the class of equations

$$u_t = \operatorname{div} (D(\Omega) \nabla u) + Q(u), \quad (7)$$

where $D \neq \text{constant}$ and Q are arbitrary smooth functions of their arguments. Equations of the form (7) also have an obvious gradient-flow interpretation.

In Section 2, a complete description of Lie symmetries of equation (7) is performed in the one-dimensional case. In Section 3, the same is done in two dimensions, the special case (2) being examined separately. It is worth emphasising at the outset that a number of the symmetries obtained are far from obvious by inspection. In Section 4, radially-symmetric reductions of equations with the power-law diffusivity to ODEs together with examples of exact solutions are presented. Finally, we discuss the results obtained in order to compare those with the known results derived for more familiar nonlinear reaction-diffusion equations.

2 Lie symmetry of a class of (1+1)-dimensional equations

In this section, we consider the class of equations

$$u_t = D(u_x)u_{xx} + Q(u), \quad (8)$$

where the smooth function $D(u_x)$ is an analog of the diffusivity in (6) (obviously (8) is equivalent to the (1+1)-dimensional equation (7) on redefining D suitably). In what follows we assume that $D(u_x) \neq \text{constant}$ because (8) with the constant diffusivity is the standard reaction-diffusion equation, for which the Lie symmetry classification was performed in [9]. It is also assumed that $Q(u) \neq \text{constant}$ because a constant source (sink) $Q(u) = q$ in (8) is removable by the substitution

$$t = t, \quad x = x, \quad u \rightarrow u - qt, \quad (9)$$

i.e. (8) with $Q(u) = q$ is equivalent to (6).

First of all we note that equation (8) with $D(u_x) = u_x^{-2}$ is linearisable by the well-known hodograph transformation. This linearisation is a direct consequence of Lie symmetry properties of (8) with $D(u_x) = u_x^{-2}$. In fact, by direct application of the Lie algorithm [17, 19] one proves the following statement (see a crucial step of the proof presented below after Theorem 3).

Theorem 1. *The nonlinear equation*

$$u_t = u_x^{-2}u_{xx} + Q(u), \quad (10)$$

with an arbitrary smooth function $Q(u)$ admits an infinite-dimensional algebra of Lie symmetries generated by the operator

$$X^\infty = w(t, u)\partial_x, \quad (11)$$

where $w(t, u)$ is an arbitrary solution of the linear PDE

$$w_t = w_{uu} - Q(u)w_u. \quad (12)$$

The hodograph substitution $x \rightarrow u$, $u \rightarrow x$ transforms (10) exactly to (12) with $w(t, u) = x(t, u)$.

Taking into account the special status of $D(u_x) = u_x^{-2}$, we consider further equation (8) with $D(u_x) \neq u_x^{-2}$. In order to obtain a complete description of Lie symmetry operators of equations belonging to class (8), we start by finding of the group \mathcal{E} of equivalence transformations as the first step. By definition, each transformation from \mathcal{E} transforms an arbitrary fixed equation of the form (26) to another equation (which can be the same equation) from this class. Nowadays finding the group \mathcal{E} for a class of PDEs is a standard procedure, which consists in application of a well-known algorithm (see e.g. [2]), hence we present the result only.

Theorem 2. *The group \mathcal{E} of the class of PDEs (8) consists of the group \mathcal{E}_c of continuous equivalence transformations generated by transformations*

$$\tilde{t} = \alpha t + \delta_0, \quad \tilde{x} = \beta x + \delta_1, \quad \tilde{u} = \gamma u + \delta_3, \quad \tilde{D} = \frac{\beta^2}{\alpha} D, \quad \tilde{Q} = \frac{\gamma}{\alpha} Q, \quad (13)$$

(here $\alpha > 0$, $\beta > 0$, $\gamma > 0$, δ_0 , δ_1 and δ_3 are arbitrary real constants) and the set of discrete transformations

1. $x \rightarrow -x$;
2. $t \rightarrow -t$, $D \rightarrow -D$, $Q \rightarrow -Q$;
3. $u \rightarrow -u$, $Q \rightarrow -Q$.

Remark 1. The discrete transformation 2 is valid under the assumption that time and diffusivity may be negative, which means that the results presented below are also valid for the associated non-physical (backward) equations.

Obviously, the discrete transformations listed above can be incorporated into the formulae (13) by allowing α , β and γ also to be negative, i.e. only the restriction $\alpha\beta\gamma \neq 0$ takes place.

Now we find so called the principal algebra (other terminology used for this algebra is the ‘trivial algebra’ and the ‘kernel of maximal invariance algebras’) of the nonlinear PDEs class (8), i.e. the maximal invariance algebra (MAI) admitted by each equation of the form (8). A direct application of the classical Lie algorithm under the assumption that the coefficients $D(u_x)$ and $Q(u)$ are arbitrary smooth functions leads the two-dimensional Abelian algebra $ALie_2$ generated by the basic operators

$$X_1 = \partial_t, \quad X_2 = \partial_x. \quad (14)$$

Table 1: Lie symmetries of the class of equations (8) with $D \neq u_x^{-2}$

	$D(u_x)$	$Q(u)$	Constraints	MAI
1.	\forall	u^{-1}		$ALie_2, X_3 = 2t\partial_t + x\partial_x + u\partial_u$
2.	\forall	u		$ALie_2, X_3 = e^t\partial_u$
3.	u_x^k	$\varepsilon_1 e^{-u}$		$ALie_2, X_3 = (k+2)t\partial_t + x\partial_x + (k+2)\partial_u$
4.	u_x^k	$\varepsilon_1 u^m$	$m \neq 1, 2$	$ALie_2, X_3 = (1-m)t\partial_t + \frac{k+1-m}{k+2}x\partial_x + u\partial_u$
5.	u_x^k	$\varepsilon_1 u^{k+1} + \varepsilon_2 u$	$k \neq \pm 1$	$ALie_2, X_3 = e^{-k\varepsilon_2 t}(\partial_t + \varepsilon_2 u\partial_u)$
6.	$\frac{1}{u_x + \gamma}$	u	$\gamma \neq 0$	$ALie_2, X_3 = e^{\varepsilon_1 t}\partial_u, X_4 = e^{\varepsilon_1 t}(\partial_t + \varepsilon_1(u + \gamma x)\partial_u)$
7.	u_x	$\varepsilon_1 u^2$		$ALie_2, X_3 = t\partial_t - u\partial_u, X_4 = t^2\partial_t - (2tu + \varepsilon_1)\partial_u$
8.	u_x	$\varepsilon_1 u^2 + \varepsilon_2$	$\varepsilon_1 \varepsilon_2 = -1$	$ALie_2, X_3 = e^{-2t}(\partial_t + 2(u - \varepsilon_1)\partial_u),$ $X_4 = e^{2t}(\partial_t - 2(u + \varepsilon_1)\partial_u)$
9.	u_x	$\varepsilon_1 u^2 + \varepsilon_2$	$\varepsilon_1 \varepsilon_2 = 1$	$ALie_2, X_3 = \cos(2t)\partial_t + 2(\sin(2t)u + \varepsilon_1 \cos(2t))\partial_u,$ $X_4 = \sin(2t)\partial_t - 2(\cos(2t)u - \varepsilon_1 \sin(2t))\partial_u$
10.	u_x^k	$\varepsilon_2 u$		$ALie_2, X_3 = x\partial_x + \left(1 + \frac{2}{k}\right)u\partial_u,$ $X_4 = e^{-k\varepsilon_2 t}(\partial_t + \varepsilon_2 u\partial_u), X_5 = e^{\varepsilon_2 t}\partial_u$

In order to obtain a complete description of Lie symmetries of the PDE class (8), one needs to find all possible pairs (D, Q) , for which the relevant nonlinear PDE is invariant under a MAI of a higher dimensionality than two.

Theorem 3. *All possible MAIs (up to equivalent transformations) of equation (8) for any fixed pair of the functions (D, Q) are presented in Table 1. Any other equation of the form (8) invariant under a non-trivial Lie algebra is reduced to one of those given in Table 1 by an equivalence transformation from \mathcal{E} .*

Remark 2. In Table 1, by rescaling and/or reflecting (see Theorem 2) the coefficients $\varepsilon_1 = \pm 1$ and $\varepsilon_2 = \pm 1$ without loss of generality. Of course, the relevant function Q in real world applications may contain arbitrarily given

non-zero coefficients λ_1 and λ_2 instead of these. This remark is valid also for Table 2 (see Section 3).

The proof. Here we apply the Lie-Ovsiannikov approach (the name of Ovsiannikov arises because he has published a remarkable paper in this direction, [18]) of the Lie symmetry classification, which is based on the classical Lie scheme and a set of equivalence transformations of the differential equation in question. It should be noted that there are some recent approaches, which are important for obtaining the so called canonical list of inequivalent equations admitting non-trivial Lie symmetry algebras and allow the solution of the problem of the Lie symmetry classification in a more efficient way than a formal application of the Lie-Ovsiannikov approach (see some remarks at the end of this section).

Thus, in order to find Lie symmetry operators, one needs to consider each equation of the form (8) (clearly this should be viewed a class of PDEs because the functions D and Q are arbitrary smooth functions) as the manifold

$$\mathcal{M} \equiv \{u_t - D(u_x)u_{xx} - Q(u) = 0\} \quad (15)$$

in the prolonged space of the variables: $t, x, u, u_t, u_x, u_{xx}, u_{xt}, u_{tt}$. According to the definition, (8) is invariant under the transformations generated by the infinitesimal operator

$$X = \xi^0(t, x, u)\partial_t + \xi^1(t, x, u)\partial_x + \eta(t, x, u)\partial_u, \quad (16)$$

if the following invariance condition is satisfied:

$$X_2(u_t - D(u_x)u_{xx} - Q(u))\Big|_{\mathcal{M}} = 0. \quad (17)$$

The operator X_2 is the second prolongation of the operator X , i.e.

$$X_2 = X + \rho_t \frac{\partial}{\partial u_t} + \rho_x \frac{\partial}{\partial u_x} + \sigma_{xx} \frac{\partial}{\partial u_{xx}} + \sigma_{tt} \frac{\partial}{\partial u_{tt}} + \sigma_{tx} \frac{\partial}{\partial u_{tx}},$$

where the coefficients ρ and σ with relevant subscripts are expressed via the functions ξ^0, ξ^1 and η by well-known formulae (see, e.g., [17, 19]).

Now we present the so called system of determining equations (DEs), obtained by direct calculations using the invariance condition (17)

$$\left(\eta_x + (\eta_u - \xi_x^1)u_x - \xi_u^1 u_x^2\right)D' + \left(\xi_t^0 - 2\xi_x^1 - 2\xi_u^1 u_x\right)D = 0, \quad (18)$$

$$-\eta_t + (\xi_t^0 - \eta_u)Q + \eta\dot{Q} + (\xi_t^1 + \xi_u^1 Q)u_x +$$

$$\left(\eta_{xx} + (2\eta_{xu} - \xi_{xx}^1)u_x + (\eta_{uu} - 2\xi_{xu}^1)u_x^2 - \xi_{uu}^1 u_x^3\right)D = 0 \quad (19)$$

(hereafter $D' = \frac{\partial D}{\partial u_x}$ and $\dot{Q} = \frac{\partial Q}{\partial u}$) together with the standard restriction $\xi^0 = \xi^0(t)$, which is common for any evolution second-order PDE.

In contrast to the system of DEs for the standard RD equation (see those in [9]), which immediately specifies the structure of the unknown functions ξ^0, ξ^1 and η , DEs (18) and (19) are much more complicated because the diffusivity D depends on u_x (other than on u). The crucial step is to solve (18) w.r.t. D , taking into account that the functions ξ^0, ξ^1 and η do not depend on u_x , so that one may formally write (18) as follows

$$(e_0 + e_1 u_x - e_2 u_x^2)D' + (e_3 - 2e_2 u_x)D = 0, \quad (20)$$

where e_k , $k = 0, \dots, 3$ correspond to the relevant coefficients in (18).

It has been noted that there exists a unique case $D = Cu_x^{-2}$ (the constant C can be removed by scaling, see Theorem 2) when the parameter e_2 can be arbitrary in (20) because only in this case does the relationship $e_2 u_x^2 D' = 2e_2 u_x D$ apply. Thus, (20) with $D = Cu_x^{-2}$ is equivalent to $e_0 = 0$, $e_3 = 2e_1$, i.e.

$$\eta_x = 0, \quad \xi_t^0 = 2\eta_u$$

while ξ^1 is still an arbitrary function. The remaining DE (19) with $D = Cu_x^{-2}$ and an arbitrary smooth function Q has the solution $\xi^1 = w(t, u)$, where $w(t, u)$ is the general solution of the linear equation (12). This means that MAI of (10) is infinite-dimensional (see operator (11)) As a result, (10) can be linearised (see Theorem 1). Thus, we assume $D \neq Cu_x^{-2}$ in what follows.

Solving the linear ODE (20), one arrives on five essentially different cases depending on the parameters.

(i) If $e_2 = e_1 = e_0 = 0$ then immediately $e_3 = 0$ and D is an arbitrary non-constant function.

(ii) If $e_2 = 0$ and $e_1 \neq 0$ then the general solutions is

$$D = C \left(u_x + \frac{e_0}{e_1} \right)^{-\frac{e_3}{e_1}}, \quad e_3 \neq 0. \quad (21)$$

(iii) If $e_2 = e_1 = 0$ and $e_0 e_3 \neq 0$ then the general solution is

$$D = C e^{-\frac{e_3}{e_0} u_x}.$$

(iv) If $e_2 \neq 0$ and $e_1^2 + e_0^2 \neq 0$ then the general solutions is

$$D = C \exp \left(- \int \frac{2e_2 u_x - e_3}{e_2 u_x^2 - e_1 u_x - e_0} du_x \right).$$

(v) If $e_2 \neq 0$, $e_1 = e_0 = 0$ and $e_3 \neq 0$ then the general solutions is

$$D = C u_x^{-2} \exp \left(- \frac{e_3}{e_0} u_x^{-1} \right).$$

Consider case (i). Having $e_3 = e_2 = e_1 = e_0 = 0$ and substituting instead of e_k , $k = 0, \dots, 3$, the relevant coefficients from (18), one arrives at a linear system of PDEs with the general solution

$$\xi^1 = \frac{1}{2} \xi_t^0 x + b(t), \quad \xi^1 = \frac{1}{2} \xi_t^0 u + p(t) \quad (22)$$

where $\xi^0(t)$, $b(t)$ and $p(t)$ are arbitrary functions. Substituting (22) into DE (19), one immediately derives a linear ODE w.r.t. $Q(u)$ because the second line of (19) simply vanishes. If $Q(u)$ is an arbitrary function then only the principal algebra (14) occurs. Non-trivial results are obtained only for $Q(u) = \lambda u^{-1}$ and $Q(u) = \lambda u$. In both cases, the arbitrary non-zero constant λ can be removed from the relevant equations using the equivalence transformations from Theorem 2. Thus, the cases 1 and 2 in Table 1 have been obtained.

Consider case (ii), which is the most complicated. Because the function D depend only on u_x , one derives from (21) the equalities $e_3 = -k e_1 \neq 0$, $e_0 = \gamma e_1$ and $e_2 = 0$ (hereafter $k \neq 0$ and γ are arbitrary constants). Taking into account the relevant coefficients from (18), one obtains the linear system of PDEs

$$\xi_t^0 - 2\xi_x^1 = -k(\eta_u - \xi_x^1), \quad \eta_x = \gamma(\eta_u - \xi_x^1), \quad \xi_u^1 = 0. \quad (23)$$

There are two essentially different subcases (iia) $\gamma \neq 0$ and (iib) $\gamma = 0$.

In subcase (iia), equations (23) immediately produce $\xi_{xx}^1 = 0$ provided $k \neq -2$ (this special exponent was checked separately; nothing interesting was found, however). Having $\xi_{xx}^1 = 0$ and equations (23), one notes that DE

(19) again reduces to a linear ODE w.r.t. $Q(u)$. Moreover, only the functions $Q(u) = \lambda u^{-1}$ and $Q(u) = \lambda u$ lead to non-trivial Lie symmetry algebras while the diffusivity is $D = C(u_x + \gamma)^k$. However, these algebras are nothing else but those arising in case 1 and 2 of Table 1 provided $k \neq -1$. The special exponent $k = -1$ was examined and the case 6 of Table 1 was found (the constants C and λ are removable using the equivalence transformations from Theorem 2).

In subcase (iib) $\gamma = 0$, equations (23) take form

$$\xi_t^0 = (2 + k)\xi_x^1 - k\eta_u, \quad \eta_x = 0, \quad \xi_u^1 = 0, \quad (24)$$

so that $\xi_{xx}^1 = 0$ ($k \neq -2$ otherwise $D = Cu_x^{-2}$). Thus, the remaining DE (19) again reduces to a linear ODE w.r.t. $Q(u)$. Its solution for arbitrary $k \neq -2$ leads to the cases 3–5 and 10 of Table 1. Moreover, it was found that case 5 is not valid provided $k = \pm 1$ (see this restriction in Table 1), because that for $k = -1$ is simply a subcase of case 4, while that for $k = 1$ leads to the very unusual Lie symmetry algebras presented in the cases 7–9 of Table 1.

Finally, it was proved by the examination of cases (iii)–(v) that there are not any further equations of the form (8), which are invariant w.r.t. a MAI of dimensionality three or higher.

The proof is now completed. ■

Thus, we have proved that there are 11 distinct cases in which equation (8) admits a non-trivial MAI, i.e. is invariant under three- or higher-dimensional Lie algebra. The case of the diffusivity $D(u_x) = u_x^{-2}$ is exceptional because the MAI is infinite-dimensional, hence, the corresponding equation is linearisable independently on the form of the function $Q(u)$. The other 10 cases are presented in Table 1. It is noteworthy that this classification is surprisingly rich: several of these Lie symmetries (see, e.g. cases 6–9) would be unlikely to be identified by using the known results for the standard reaction-diffusion equation [9].

Finally in this context, we present the following observation. The substitution

$$\tau = \frac{1}{\varepsilon_2 k} e^{\varepsilon_2 k t}, \quad x = x, \quad w = e^{-\varepsilon_2 t} u \quad (25)$$

reduces the equations listed in cases 5 and 10 to the equation with $m = k + 1$ listed in case 4 and to the equation $w_\tau = w_x^k w_{xx}$, respectively. Thus, Table 1

can be shortened to 8 cases. However, substitution (25) does not belong to the group \mathcal{E} , hence we have found so called form-preserving transformations [16] (it should be noted that this kind of transformations was introduced earlier in [15] but without above terminology) allowing a reduction of the number of cases obtained via the classical Lie-Ovsiannikov algorithm (see extensive discussions in [6], [7], [8] on this matter).

3 Lie symmetry of a class of (1+2)-dimensional equations

Here we consider a class of equations of the form

$$u_t = \operatorname{div} (D(\Omega) \nabla u) + Q(u), \quad \Omega = |\nabla u|^2, \quad (26)$$

where $u = u(t, x_1, x_2) \in \mathbb{R}$, $t \in \mathbb{R}$, $(x_1, x_2) \in \mathbb{R}^2$; $D(\Omega)$ and $Q(u)$ are arbitrary smooth functions of their arguments. It can be easily noted that equation (26) with $Q(u) = q \equiv \text{constant}$ again reduces to (2), by the substitution

$$t = t, \quad x_1 = x_1, \quad x_2 = x_2, \quad u \rightarrow u - qt, \quad (27)$$

so that we assume in what follows that either $Q(u)$ is non-constant or $Q(u) = 0$.

3.1 The case of a non-constant source

We start from the general case when $Q(u)$ is an arbitrary non-constant function. Similarly to the (1+1)-dimensional case, we find the group \mathcal{E} of equivalence transformations as the first step.

Theorem 4. *The group \mathcal{E}_c of continuous equivalence transformations of the class of PDEs (26) consists of the transformations*

$$\tilde{t} = \alpha t + \delta_0, \quad \tilde{x}_a = \beta \gamma_{ab} x_b + \delta_a, \quad \tilde{u} = \gamma u + \delta_3, \quad \tilde{D} = \frac{\beta^2}{\alpha} D, \quad \tilde{Q} = \frac{\gamma}{\alpha} Q, \quad (28)$$

where $\alpha, \beta, \gamma, \delta_k$ ($k = 0, \dots, 3$), γ_{ab} ($a, b = 1, 2$) are arbitrary real constants satisfying the conditions $\alpha > 0$, $\beta > 0$, $\gamma > 0$, and the matrix $(\gamma_{ab}) \in SO(2)$.

It can be noted that the set of discrete transformations

1. $x_1 \rightarrow -x_1, x_2 \rightarrow -x_2$;
2. $t \rightarrow -t, D \rightarrow -D, Q \rightarrow -Q$;
3. $u \rightarrow -u, Q \rightarrow -Q$.

also belongs to the group \mathcal{E} , hence, one may claim that the group of equivalence transformations of the nonlinear PDEs class (26) consists of the transformations defined by formulas (28) with arbitrary real non-zero parameters α, β , and γ .

As the second step, we find the principal algebra i.e. the MAI admitted by each equation of the form (26), which comprises the operators of translations and rotations.

Theorem 5. *The principal algebra of the PDEs class (26) is the four-dimensional Lie algebra $ALie_4$ generated by the basic operators*

$$X_1 = \partial_t, X_2 = \partial_{x_1}, X_3 = \partial_{x_2}, X_4 = x_2 \partial_{x_1} - x_1 \partial_{x_2}.$$

The proof of this theorem consists in the direct application of the classical Lie algorithm under the assumption that the coefficients $D(\Omega)$ and $Q(u)$ are arbitrary smooth functions. The relevant calculations are omitted here because they are simple but cumbersome and the result (namely the operators of translations and rotations) is to be expected.

The third and final step is highly nontrivial: to find all pairs (D, Q) , when the relevant nonlinear PDE (26) is invariant under a MAI of a higher dimensionality than four. Note that we present the result obtained in two tables because the diffusion coefficient $D = \Omega^{-1}$ is special and this case, which corresponds to a limit case of the second formula in (4), is especially noteworthy.

Theorem 6. *All possible MAIs (up to the equivalence transformations) of equation (26) for any fixed pair of the function (D, Q) are presented in Tables 2 and 3, where the designations $\varepsilon_1 = \pm 1, \varepsilon_2 = \pm 1$ are used and $k \neq 0, \lambda \neq 0, m$ are arbitrary real constants. Any other equation of the form (26) invariant under a non-trivial Lie algebra is reduced to one of those given in Tables 2 and 3 by an equivalence transformation from the group \mathcal{E} .*

Remark 3. Similarly to the (1+1)-dimensional case, the substitution

$$\tau = \frac{1}{2\varepsilon_2 k} e^{2\varepsilon_2 k t}, x_1 = x_1, x_2 = x_2, w = e^{-\varepsilon_2 t} u \quad (29)$$

Table 2: Lie symmetries of the class of equations (26): $D \neq \Omega^{-1}$

	$D(\Omega)$	$Q(u)$	Constraints	MAI
1.	\forall	u^{-1}		$ALie_4, X_5 = 2t\partial_t + x_1\partial_{x_1} + x_2\partial_{x_2} + u\partial_u$
2.	\forall	u		$ALie_4, X_5 = e^t\partial_u$
3.	Ω^k	$\varepsilon_1 e^{-u}$		$ALie_4, X_5 = 2(k+1)t\partial_t + x_1\partial_{x_1} + x_2\partial_{x_2} + 2(k+1)\partial_u$
4.	Ω^k	$\varepsilon_1 u^m$	$m \neq 0, 1, 2$	$ALie_4, X_5 = 2t\partial_t + \frac{m-2k-1}{m-2k-1+k(m+1)} \times (x_1\partial_{x_1} + x_2\partial_{x_2}) - \frac{2(k+1)}{m-2k-1+k(m+1)} u\partial_u$
5.	Ω^k	$\varepsilon_1 u^{2k+1} + \varepsilon_2 u$	$k \neq -\frac{1}{2}, \frac{1}{2}$	$ALie_4, X_5 = e^{-2k\varepsilon_2 t}(\partial_t + \varepsilon_2 u\partial_u)$
6.	$\Omega^{\frac{1}{2}}$	$\varepsilon_1 u^2$		$ALie_4, X_5 = t\partial_t - u\partial_u, X_6 = t^2\partial_t - (2tu + \varepsilon_1)\partial_u$
7.	$\Omega^{\frac{1}{2}}$	$\varepsilon_1 u^2 + \varepsilon_2$	$\varepsilon_1 \varepsilon_2 = -1$	$ALie_4, X_5 = e^{-2t}(\partial_t + 2(u - \varepsilon_1)\partial_u),$ $X_6 = e^{2t}(\partial_t - 2(u + \varepsilon_1)\partial_u)$
8.	$\Omega^{\frac{1}{2}}$	$\varepsilon_1 u^2 + \varepsilon_2$	$\varepsilon_1 \varepsilon_2 = 1$	$ALie_4, X_5 = \cos(2t)\partial_t + 2(\sin(2t)u + \varepsilon_1 \cos(2t))\partial_u,$ $X_6 = \sin(2t)\partial_t - 2(\cos(2t)u - \varepsilon_1 \sin(2t))\partial_u$
9.	Ω^k	$\varepsilon_2 u$		$ALie_4, X_5 = x_1\partial_{x_1} + x_2\partial_{x_2} + (1 + \frac{1}{k})u\partial_u,$ $X_6 = e^{-2k\varepsilon_2 t}(\partial_t + \varepsilon_2 u\partial_u), X_7 = e^{\varepsilon_2 t}\partial_u$

 Table 3: Lie symmetries of the class of equations (26): $D = \Omega^{-1}$

	$Q(u)$	MAI
1.	\forall	$ALie_4, X_5 = x_1\partial_{x_1} + x_2\partial_{x_2}$
2.	$\lambda u^{-1} + \varepsilon_2 u$	$ALie_4, X_5 = x_1\partial_{x_1} + x_2\partial_{x_2}, X_6 = e^{2\varepsilon_2 t}(\partial_t + \varepsilon_2 u\partial_u)$
3.	λu^{-1}	$ALie_4, X_5 = x_1\partial_{x_1} + x_2\partial_{x_2}, X_6 = 2t\partial_t + u\partial_u$
4.	λu	$ALie_4, X_5 = x_1\partial_{x_1} + x_2\partial_{x_2}$ $X_6 = e^{2\lambda t}(\partial_t + \lambda u\partial_u), X_7 = e^{\lambda t}\partial_u$

reduces the equations listed in cases 5 and 9 to the equation with $m = 2k + 1$ listed in case 4 and to the equation $w_\tau = \operatorname{div}(|\nabla w|^{2k}\nabla w)$, respectively.

Remark 4. Substitution (29) with $k = -1$ reduces the equations listed in cases 2 and 4 of Table 3 to the equation listed in case 3 and to the equation $w_\tau = \operatorname{div}(|\nabla w|^{-2}\nabla w)$, respectively. Thus, Table 3 consists of two essentially different cases only; however, both cases have no analogs in Table 2.

3.2 Nonlinear diffusion equation with gradient-dependent diffusivity

It is quite natural to present the result for the nonlinear equation (2) (hereafter with $n = 2$) separately because of its importance for direct applications.

Theorem 7. *The group \mathcal{E} of equivalence transformations of the class of equations (2) consists of the group of continuous equivalence transformations*

$$\tilde{t} = \alpha t + \delta_0, \quad \tilde{x}_a = \beta \gamma_{ab} x_b + \delta_a, \quad \tilde{u} = \gamma u + \delta_3, \quad \tilde{D} = \frac{\beta^2}{\alpha} D, \quad (30)$$

(the restrictions on the parameters are the same as in (28)) and the discrete transformations

1. $x_1 \rightarrow -x_1$;
2. $t \rightarrow -t, D \rightarrow -D$;
3. $u \rightarrow -u$.

Similarly to the general class (26), the group of equivalence transformations of (2) is represented only by the formulae (30), though with arbitrary non-zero parameters α , β , and γ .

In comparison to the principal algebra of the PDE class (26), that of (2) is more complicated but still can be easily calculated by direct application of the Lie algorithm, hence the following statement has been proved.

Theorem 8. *The principal algebra of (2) is the six-dimensional Lie algebra $ALie_6$ generated by the basic operators*

$$X_1, X_2, X_3, X_4, X_5 = \partial_u, X_6 = 2t\partial_t + x_1\partial_{x_1} + x_2\partial_{x_2} + u\partial_u \quad (31)$$

(see X_1, X_2, X_3 , and X_4 in Theorem 5).

Remark 5. The Lie algebra $ALie_6$ is a semidirect sum of the extended Euclidean algebra with the basic operators X_2, X_3, X_4, X_6 and the Abelian algebra with the basic operators X_1, X_5 .

As the final step, we perform an analysis in order to find all possible functions $D(\Omega)$ leading to extensions of the Lie algebra $ALie_6$. It turns out that the result is (perhaps surprisingly) rather simple.

Theorem 9. *Equation (2) admits a MAI of a higher dimensionality than six if and only if the function $D(\Omega) = \Omega^k \equiv |\nabla u|^{2k}$, $k \neq 0$ (up to the equivalence transformations from the group \mathcal{E}). The relevant MAI is seven-dimensional and generated by the operators (31) and the operator*

$$X_7 = 2kt\partial_t - u\partial_u. \quad (32)$$

Corollary 1. *Equation (2) with the diffusion coefficients D of the form (4) is invariant with respect to the MAI with the basic operators (31).*

4 Reductions of the nonlinear equation (2) to ODEs

According to Theorem 9, equation (2) possesses its widest Lie symmetry in the case of the power-law diffusivity. Moreover, this case is important from the potential applicability point of view (see, e.g. [4]). Thus, here we examine the nonlinear (1+2)-dimensional equation

$$u_t = \operatorname{div}(|\nabla u|^{2k} \nabla u). \quad (33)$$

Using the seven-dimensional Lie algebra with the basic operators (31)-(32) a wide range reductions of (33) to (1+1)-dimensional PDEs and to ODEs can be obtained. Here we restrict ourselves to an important case when (33) describes a process with radial symmetry w.r.t. to space variables (compare for instance an example in [20]). This allows to reduce equation (33) to the form

$$U_t = \frac{1}{r} (r U_r^{2k+1})_r \quad (34)$$

(here we assume $U_r > 0$, while the case $U_r < 0$ leads to $U_t = -\frac{1}{r} (r(-U_r)^{2k+1})_r$, which can be treated in the same way) by the ansatz

$$u(t, x_1, x_2) = U(t, r), \quad r = \sqrt{x_1^2 + x_2^2} \quad (35)$$

generated by the operator of rotations $X_4 = x_2\partial_{x_1} - x_1\partial_{x_2}$.

First of all we note that equation (34) with $k = -1$, i.e.

$$U_t = \frac{1}{r} (rU_r^{-1})_r \quad (36)$$

is transformed to the linear equation

$$V_t = -V_{zz} \quad (37)$$

by the hodograph type substitution

$$r = \sqrt{V}, \quad U = z. \quad (38)$$

Thus, we assume in what follows that $k \neq -1$. Moreover, we assume that $k \neq 0$ (otherwise the linear radially-symmetric heat equation is obtained) and $k \neq -\frac{1}{2}$ because equation (34) degenerates to a linear ODE in this case.

Remark 6. In order to describe a diffusion process equation (34) should be taken with the sign of $2k+1$ on the RHS. Especially taking $k = -1$ one arrives at the linear heat equation instead of (37). However, this observation does not affect any results presented below because both equations are equivalent up to the discrete transformation $t \rightarrow -t$.

It can be easily calculated that the MAI of (34) is four-dimensional Lie algebra with the basic operators

$$X_0 = \partial_U, \quad X_1 = \partial_t, \quad D_0 = 2(k+1)t\partial_t + r\partial_r, \quad D_1 = kr\partial_r + (k+1)U\partial_U \quad (39)$$

i.e. the reduction of (33) to the radial form partly inherits the relevant symmetries of this equation (see (31) and (32)) and no others. A simple analysis shows that this Lie algebra leads to four essential different reductions of (34) to ODEs (any other reduction is a composition of one of these four and the relevant group of transformations generated by Lie algebra (39)). Four essentially different combinations of the operators from (39), the relevant ansätze and ODEs are as follows.

Case (i) $X_1, U = \phi(r)$. ODE for $\phi(r)$:

$$(r\phi_r^{2k+1})_r = 0. \quad (40)$$

Case (ii) $D_0 + \lambda X_0, U = \frac{\lambda}{2(k+1)} \log t + \phi(\omega), \omega = rt^{-\frac{1}{2(k+1)}}$. ODE for $\phi(\omega)$:

$$(2k+1)\phi'' + \omega^{-1}\phi' + \frac{1}{2(k+1)}(\phi')^{1-2k} - \frac{\lambda}{2(k+1)}(\phi')^{-2k} = 0. \quad (41)$$

Case (iii) $D_1 + \lambda X_1$, $U = e^{\frac{(k+1)t}{\lambda}} \phi(\omega)$, $\omega = r e^{-\frac{kt}{\lambda}}$, $\lambda \neq 0$. ODE for $\phi(\omega)$:

$$(2k+1)\phi'' + \omega^{-1}\phi' + \frac{k}{\lambda}\omega(\phi')^{1-2k} - \frac{k+1}{\lambda}\phi(\phi')^{-2k} = 0. \quad (42)$$

The subcase with $\lambda = 0$ leads to the separable ansatz $U = r^{\frac{k+1}{k}} \phi(t)$, which produces the first-order ODE

$$\phi' = \left(\frac{3k+1}{k}\right) \left(\frac{k+1}{k}\right)^{1+2k} \phi^{1+2k}. \quad (43)$$

Case (iv) $D_0 + \lambda D_1$, $\lambda \neq 0$, $U = t^{\frac{\lambda}{2}} \phi(\omega)$, $\omega = r t^{-\gamma}$, $\gamma = \frac{1+\lambda k}{2(k+1)}$. ODE for $\phi(\omega)$:

$$(2k+1)\phi'' + \omega^{-1}\phi' + \gamma\omega(\phi')^{1-2k} - \frac{\lambda}{2}\phi(\phi')^{-2k} = 0. \quad (44)$$

It is a simple task to integrate the nonlinear ODEs (40) and (43) because of their simplicity and we omit this. The nonlinear ODEs (41), (42) and (44) are not integrable for arbitrary parameters (i.e. similarity exponents).

ODE (41) is equivalent to the first-order ODE

$$(2k+1)y' + \omega^{-1}y + \frac{1}{2(k+1)}y^{1-2k} - \frac{\lambda}{2(k+1)}y^{-2k} = 0, \quad (45)$$

where $y(\omega) = \phi'$. To the best of our knowledge the general solution of ODE (45) for arbitrary parameters k and λ is unknown (at least in terms of elementary functions). However, we note that this equation for $\lambda = 0$ is a Bernoulli ODE and can hence be linearised. Hence the general solution of ODE (41) with $\lambda = 0$ of the form

$$\phi = \int \left(\frac{-k\omega^2 + C_1\omega^{-2k/(2k+1)}}{2(k+1)(3k+1)} \right)^{\frac{1}{2k}} d\omega + C_2 \quad (46)$$

is constructed (hereafter C_1 and C_2 are arbitrary constants). Thus, one obtains a two-parameter family of exact solutions for the nonlinear (1+2)-dimensional equation (33) of the form

$$u = \int \left(\frac{-k\omega^2 + C_1\omega^{-2k/(2k+1)}}{2(k+1)(3k+1)} \right)^{\frac{1}{2k}} d\omega + C_2, \quad \omega = t^{-\frac{1}{2(k+1)}} \sqrt{x_1^2 + x_2^2}. \quad (47)$$

Obviously, the integral in the RHS of (47) can be in terms of elementary functions in some cases, for example, for $C_1 = 0$ and for any $C_1 \neq 0$ provided $k = \frac{1}{2}, \frac{1}{4}, \frac{1}{6} \dots$.

We next consider ODE (42), which, unlike to the previous case, cannot be reduced to a first-order ODE for arbitrary parameters k and λ . However, this equation can be rewritten in the form

$$\left(\omega(\phi')^{1+2k}\right)' = \frac{k+1}{\lambda}\omega\phi - \frac{k}{\lambda}\omega^2\phi'$$

and one may note the special case $k+1 = -2k$, i.e. $k = -\frac{1}{3}$, when this equation is equivalent to the first-order ODE

$$\omega(\phi')^{\frac{1}{3}} = \frac{1}{3\lambda}\omega^2\phi + C_1.$$

This ODE with $C_1 \neq 0$ has no solutions in terms of elementary functions while setting $C_1 = 0$ one easily finds

$$\phi = \pm \frac{2}{\sqrt{C_2 - 2(3\lambda)^{-3}\omega^4}}. \quad (48)$$

Thus, one obtains exact solutions of the nonlinear (1+2)-dimensional equation (33) with $k = -\frac{1}{3}$ of the form

$$u = \pm \frac{2}{\sqrt{C_2 e^{-\frac{2t}{3\lambda}} - 2(3\lambda)^{-3}(x_1^2 + x_2^2)^2}}. \quad (49)$$

Finally we turn to the nonlinear ODE (44). A detailed analysis of this equation was undertaken in [4] (see equation (3.6) with $k = 1$ therein). In particular, it was noted that there is special case $\gamma = -\frac{\lambda}{4}$, i.e., $\lambda = -\frac{2}{3k+1}$ (see (44) above) when this equation can be reduced to the first-order ODE

$$\omega(\phi')^{2k+1} = \frac{1}{2(3k+1)}\omega^2\phi + C_1$$

in the same way as we have done above for ODE (42). The exact solution obtained (see formulae (2.3) in [4]) was used for solving the Cauchy problem with the Dirac function as the initial profile.

5 Discussion

In this paper, complete descriptions of Lie symmetries of equation (7) in 1-D and 2-D spaces (i.e. one and two space variables, respectively) have been obtained. As one may note, the results depend essentially on the space dimensionality of (7) (so that 3-D and higher-dimensionality cases should be also examined) and on the form of the pair (D, Q) . We have established a special case (see Theorem 1) when this equation in 1-D space possesses an infinite-dimensional Lie algebra of invariance so that it is linearisable. We also have found the special diffusivity $D = |\nabla u|^{-2}$, which is special one of (7) in 2-D space (see Table 3). It should be stressed that equation (7) depending on the form of (D, Q) admits a surprisingly rich set of Lie symmetry algebras (see Tables 1 and 2). Several of these Lie symmetries would be unlikely to be identified by a simple extrapolation from the known results [9, 10] for the standard reaction-diffusion equation

$$u_t = \operatorname{div}(D(u)\nabla u) + Q(u).$$

We have examined separately equation (2) with $n = 2$, which sometimes is called the Perona-Malik equation, because of its importance for direct applications. In contrast to the (1+2)-dimensional nonlinear diffusion equation (1), i.e.

$$u_t = \operatorname{div}(D(u)\nabla u),$$

which admits a wide range of MAIs depending on the form the diffusivity $D(u)$ [12], this alternative equation admits a unique extension of the MAI (see Theorem 9). In this sense equation (2) is not analogous to the standard nonlinear diffusion equation (1) (at least from the Lie symmetry point of view). Moreover, equation (2) with the common nonlinearities (4) does not lead to any extension of the Lie symmetry. We believe it plausible that a search for conditional symmetries of the nonlinear (1+2)-dimensional equation (2) in order to compare with the known result for the (1+2)-dimensional nonlinear diffusion equation (1) [3] could lead to more optimistic results.

On the other hand, one may consider the (1+2)-dimensional equation

$$u_t = D(\Omega)\nabla^2 u, \tag{50}$$

instead of (2). In contrast to the (1+1)-dimensional case, equation (50) is not equivalent to (2). On the other hand, this equation possesses an infinite-dimensional Lie algebra in the case $D(\Omega) = \Omega^{-1}$. Moreover this algebra

contains the operators of the form

$$X^\infty = A(x_1, x_2)\partial_{x_1} + B(x_1, x_2)\partial_{x_2}$$

where the pair $(A(x_1, x_2), B(x_1, x_2))$ is an arbitrary solution of the Cauchy-Riemann system, having similar structure to the operators

$$X^\infty = A(x_1, x_2)\partial_{x_1} + B(x_1, x_2)\partial_{x_2} - 2A_{x_1}u\partial_u$$

that arise in the MAI for the nonlinear diffusion equation (1) with $D(u) = u^{-1}$. Thus, investigation of Lie symmetry properties of equation (50) in higher dimensions may lead to essentially different results comparing with (2).

Finally, we have reduced the (1+2)-dimensional equation (2) with a power-law diffusivity, i.e. equation (33), to a corresponding (1+1)-dimensional equation in order to find exact solutions. It turns out that the equation obtained still possesses a rich symmetry, hence further reductions to ODEs have been performed. As a result, two families of new exact solutions of equation (33) have been constructed (see formulae (47) and (49)) and solutions found much earlier in [4] have been recovered.

6 Acknowledgements

This research was supported by a Marie Curie International Incoming Fellowship to the first author within the 7th European Community Framework Programme.

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